Ergodicity and stability of hybrid systems with threshold type state-dependent switching

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Based on the joint work with J.Shao & Q.W

@Workshop on Markov Processes and Related Topics, 2022.11.25–11.27

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Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a complete probality. Consider $(X_t, \Lambda_t)_{t\geq 0}$ as follows:

$$\begin{cases} \mathrm{d}X_t = b(X_t, \Lambda_t)\mathrm{d}t + \sigma(X_t, \Lambda_t)\mathrm{d}B_t, & t \geq 0, \\ (X_0, \Lambda_0) = (x_0, i_0), \end{cases} \tag{1}$$

- ♠ $\mathbb{S} = \{1, 2, \cdots N\}$, $X_t \in \mathbb{R}^d$, $b(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}^d$, $\sigma(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}^d$, (B_t) is a BM on $\mathbb{R}^d$, independent of $\{\Lambda_t\}_{t\geq 0}$.
- ♠ $\{\Lambda_t\}_{t\geq 0}$ has state space $\mathbb{S}$, and its transition matrix satisfies

$$\mathbb{P}(\Lambda_{t+\delta} = j | \Lambda_t = i, X_t = x) = \begin{cases} q_{ij}(x)\delta + o(\delta), & i \neq j, \\ 1 + q_{ii}(x)\delta + o(\delta), & i = j, \end{cases} \tag{2}$$

- ♠ Assumption: $Q(x) = (q_{ij}(x))$ is irreducible and conservative.

Applications in mathematical finance, biology et.al.

- X. Guo, Q. Zhang(2015), perpetual American put options ; L. Sotomayor, A. Cadenillas(2009), consumption-investment problems in financial markets;
- J. Fontbona, H. Guérin, F. Malrieu(2012), Quantitative estimates for the long-time behavior of an ergodic variant of the telegraph process; A. Crudu, A. Bebussche, A. Muller, O. Radulescu(2012), gene networks to hybrid piecewise deterministic processes.

Long time behavior of such processes

- The exponential ergodicity:

  in the total variance distance: M. Pinsky, R. Pinsky(1993), J. Shao(2015)

  in the Wasserstein distance: B. Cloez, M. Hairer(2015), J. Shao(2015)

Introduction–Background

- Stability in various sense:

  Monographs: X. Mao, C. Yuan(2006), G. Yin, C. Zhu(2010),
  Literatures: J. Shao, F. Xi(2014), G. Basak, A. Bisi, M. Ghosh(1999)

- The characterization of the invariant probability measures:

  J. Bardet, H. Guerin, F. Malrieu(2010), B. de Saporta, J.-F. Yao(2005),Z. Liao, J.Shao(2020), S. Q, Zhang(2019)

- More:

  Numerical solutions, General decay rates,$\cdots\cdots$
  F.K, Wu; Q.X, Zhu; G.Q, Lan; Chatterjee, Liberzon, $\cdots\cdots$

Introduction–Background

Two kinds of methods have been developed to deal with the state-dependent regime-switching processes.

- Construct suitable coupling process to control the state-dependent regime-switching process with a state-independent one.

  B. Cloez, M. Hairer(2015); A. Majda, X. Tong(2016); J.Shao(2018, 2022); J. Shao, F. Xi(2019).

- Construct directly the desired Lyapunov function by viewing the whole system as a Markov process;

  Monograph: G. Yin, C. Zhu(2010); J,Shao(2015), J. Shao, F. Xi(2014) based on M-matrix theory.;

These two methods could provide certain verifiable conditions at the cost of sharpness. In this work, we shall develop an alternative method: approximation method.

Introduction–Background

- Approximate a general continuous matrix-valued function with a sequence of step functions, i.e. piecewise constant matrix-valued functions.
- The approximation is measured using the Wasserstein distance between the distributions of two processes, and the convergence rate can also be estimated in terms of the distance between the transition rate matrices.
- As a special class of state-dependent regime-switching processes, we can provide a characterization on the ergodicity and stability for stochastic hybrid systems with threshold type switching.

For $m \in \mathbb{N}$, $(q_{ij}^{(k)})_{i,j \in \mathcal{S}}$ are Q-matrices $k = 1, \ldots, m+1$.

$d \geq 2$: Let $\Delta_m := \{0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m < \alpha_{m+1} = \infty\}$, a finite division of $[0, \infty)$,

$$q_{ij}(x) = \sum_{k=1}^{m+1} q_{ij}^{(k)} \mathbf{1}_{[\alpha_{k-1}, \alpha_k)}(|x|), \qquad i, j \in \mathcal{S},\ x \in \mathbb{R}^d, \tag{3}$$

$d = 1$: Let $\{-\infty < \alpha_0 < \alpha_1 < \ldots < \alpha_m < \alpha_{m+1} = \infty\}$ is a finite partition of $\mathbb{R}$.

$$q_{ij}(x) = q_{ij}^{(0)} \mathbf{1}_{(-\infty, \alpha_0)}(x) + \sum_{k=1}^{m+1} q_{ij}^{(k)} \mathbf{1}_{[\alpha_{k-1}, \alpha_k)}(x),\ i, j \in \mathcal{S},\ x \in \mathbb{R}, \tag{4}$$

♠ We call stochastic hybrid system $(X_t, \Lambda_t)_{t \geq 0}$ with $(\Lambda_t)_{t \geq 0}$ satisfying (2), (3) or (34) when $d = 1$ a *stochastic hybrid system with threshold type switching.*

Motivation

- The main reason to study the switching function in the form (3) is its simplicity
- such functions are widely used in various research fields.
  - J. Macki, A. Strauss(1982), J.M. Harrison, M.I. Taksar(1983): bang-bang policy.
  - In the study of particle system, Cox and Durrett(1992) discovered that certain nonlinear voter models can coexist. Among those of greatest interest are the threshold voter models (cf. T. Liggett(1994)).
  - See R. Durrett(1995) for more threshold models; R. Durrett, S. Levin(1994) for the biological applications of these models of interacting particle system.
- However, the hybrid system with switching rates being a step function in x has not been studied before.

Aim-1

Provide conditions to ensure the wellposedness of hybrid system with threshold type switching

- Generalize the Skorokhod representation theorem to deal with the non-continuity of $x \mapsto q_{ij}(x)$.
- M. Ghosh, A. Arapostathis, S. Marcus(1993); J. Shao(2015); G. Yin, C. Zhu(2010).

Assumption A:

(A1) There exists $K_1 > 0$ such that

$$|b(x,i)-b(y,i)|+\|\sigma(x,i)-\sigma(y,i)\| \le K_1|x-y|, x,y \in \mathbb{R}^d, \ i \in \mathcal{S}.$$

(A2) For each $k \ge 0$, the Q-matrix $(q_{ij}^{(k)})_{i,j\in\mathcal{S}}$ in (3) or (34) is irreducible and conservative, which means that $q_i^{(k)} = -q_{ii}^{(k)} = \sum_{j\neq i} q_{ij}^{(k)}$ for every $i \in \mathcal{S}$.

- Condition (A1) is used to ensure the existence and uniqueness of the solution to (1) as usual, which can be weakened to be non-Lipschitz as in Shao(2015), or be in certain integrable space as in S. Q. Zhang(2019).
- Condition (A2) is a standard condition in the study of continuous time Markov chains.

Main results–Aim-1

- Classical: Construct consecutive left closed right-open intervals according to lexicographic ordering on $\mathbb{S} \times \mathbb{S}$.
- $K_0 = \max\left\{q_i^{(k)};\ i \in \mathcal{S}, 1 \le k \le m+1\right\}$.

  $\Gamma_{1k}(x) = [(k-2)K_0, (k-2)K_0 + q_{1k}(x)), \quad 2 \le k \le N,\ U_1 = [0, NK_0)$.
- For $n \ge 2$, When $k < n$,

  $\Gamma_{nk}(x) = \big[2(n-1)NK_0 - (n-k)K_0, 2(n-1)NK_0 - (n-k)K_0 + q_{nk}(x)\big)$,

  When $k > n$,

  $\Gamma_{nk}(x) = \big[2(n-1)NK_0 + (k-n-1)K_0, 2(n-1)NK_0 + (k-n-1)K_0 + q_{nk}(x)\big)$

  $U_n = \big[(2n-3)NK_0, (2n-1)NK_0\big)$, $n \ge 2$. $\Gamma_{nk}(x) \subset U_n$.
- Let $\kappa_0 = (2N-1)NK_0$. then $U_n \subset [0, \kappa_0]$ for $1 \le n \le N$. $\Gamma_{ii}(x) = \emptyset$ and $\Gamma_{ij}(x) = \emptyset$ if $q_{ij}(x) = 0$.
- Let

$$\vartheta(x, i, z) = \begin{cases} j - i, & \text{if } z \in \Gamma_{ij}(x), \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Let us consider the SDEs

$$\mathrm{d}\widetilde{X}_t = b(\widetilde{X}_t, \widetilde{\Lambda}_t)\mathrm{d}t + \sigma(\widetilde{X}_t, \widetilde{\Lambda}_t)\mathrm{d}B_t, \tag{6}$$

$$\mathrm{d}\widetilde{\Lambda}_t = \int_{[0,\kappa_0]} \vartheta(\widetilde{X}_t, \widetilde{\Lambda}_{t-}, z)\mathcal{N}(\mathrm{d}t, \mathrm{d}z) \tag{7}$$

with initial value $(\widetilde{X}_0, \widetilde{\Lambda}_0) = (x_0, i_0) \in \mathbb{R}^d \times \mathcal{S}$, where $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion; $\mathcal{N}(\mathrm{d}t, \mathrm{d}z)$ is a Poisson random measure over $[0, \kappa_0]$ with intensity measure $\mathrm{d}t \times \mathrm{d}z$ and independent of $(B_t)_{t\geq 0}$.

Recall (3): $\Delta_m := \{0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m < \alpha_{m+1} = \infty\}$, a finite division of $[0, \infty)$,

$$q_{ij}(x) = \sum_{k=1}^{m+1} q_{ij}^{(k)} \mathbf{1}_{[\alpha_{k-1}, \alpha_k)}(|x|), \qquad i, j \in \mathcal{S},\ x \in \mathbb{R}^d.$$

Theorem1

Assume (A1) and (A2) hold. Then the system of SDEs (6), (7) admits a pathwise unique strong solution $(\widetilde{X}_t, \widetilde{\Lambda}_t)_{t \geq 0}$ for every initial value $(x_0, i_0) \in \mathbb{R}^d \times \mathcal{S}$.
Assume, in addition, that for every $t \geq 0$, $\mathbb{P}(|\widetilde{X}_t| = \alpha_k) = 0$ for $k = 0, \ldots, m$, then $(\widetilde{X}_t, \widetilde{\Lambda}_t)$ is a solution to (1), (2) with $(q_{ij}(x))$ satisfying (3).

Aim–2

Approximation problem of hybrid systems with threshold type switching

If we use a sequence of state-dependent Q-matrix in the form (3) to approximate a state-dependent Q-matrix $(q_{ij}(x))$ satisfying $x \mapsto q_{ij}(x)$ being Lipschitz continuous, the corresponding hybrid systems will converge to the limit system.

Main results–Aim-2

- $Q(x) = (q_{ij}(x))_{i,j\in\mathcal{S}}$ be a conservative, irreducible Q-matrix on $\mathcal{S}$ for every $x \in \mathbb{R}^d$. $x \mapsto q_{ij}(x)$ is continuous, $q_i(x) = -q_{ii}(x)$. $\tilde{\kappa}_0 := \sup_{x\in\mathbb{R}^d} \max_{i\in\mathcal{S}} q_i(x) < \infty$;
- Q-matrices $Q^{(n)}(x) = (q_{ij}^{(n)}(x))_{i,j\in\mathcal{S}}$: $(q_{ij}^{n,k})_{i,j\in\mathcal{S}}$ is a conservative, irreducible Q-matrix on $\mathcal{S}$ for every $n \geq 1$, $k = 1, \ldots, m_n$.

$$\Delta_{m_n}^n := \{0 = \alpha_0^n < \alpha_1^n < \ldots < \alpha_{m_n}^n < \alpha_{m_n+1}^n = +\infty\},$$

$$q_{ij}^{(n)}(x) = \sum_{k=1}^{m_n+1} q_{ij}^{n,k} \mathbf{1}_{[\alpha_{k-1}^n, \alpha_k^n)}(|x|) \tag{8}$$

- $\Theta_n := \sup_{x\in\mathbb{R}^d} \|Q^{(n)}(x) - Q(x)\|_{\ell_1} = \sup_{x\in\mathbb{R}^d} \max_{i\in\mathcal{S}} \sum_{j\neq i} |q_{ij}^{(n)}(x) - q_{ij}(x)| \to 0, n \to \infty.$

$(X_t, \Lambda_t)_{t\geq 0}$ can be expressed as a solution to the SDE

$$\begin{cases} \mathrm{d}X_t = b(X_t, \Lambda_t)\mathrm{d}t + \sigma(X_t, \Lambda_t)\mathrm{d}B_t, \\ \mathrm{d}\Lambda_t = \int_{[0,\kappa_1]} \vartheta(X_t, \Lambda_{t-}, z)\mathcal{N}(\mathrm{d}t, \mathrm{d}z) \end{cases} \tag{9}$$

with $(X_0, \Lambda_0) = (x_0, i_0) \in \mathbb{R}^d \times \mathcal{S}$. $(X_t^{(n)}, \Lambda_t^{(n)})_{t\geq 0}$ can be expressed as a solution to the SDE

$$\begin{cases} \mathrm{d}X_t^{(n)} = b(X_t^{(n)}, \Lambda_t^{(n)})\mathrm{d}t + \sigma(X_t^{(n)}, \Lambda_t^{(n)})\mathrm{d}B_t, \\ \mathrm{d}\Lambda_t^{(n)} = \int_{[0,\kappa_1]} \vartheta^{(n)}(X_t^{(n)}, \Lambda_{t-}^{(n)}, z)\mathcal{N}(\mathrm{d}t, \mathrm{d}z) \end{cases} \tag{10}$$

with $(X_0^{(n)}, \Lambda_0^{(n)}) = (x_0, i_0) \in \mathbb{R}^d \times \mathcal{S}$, $\vartheta(x, i, z)\vartheta^{(n)}(x, i, z)$ are defined as in (5) associated with $(q_{ij}(x))_{i,j\in\mathcal{S}}$,$\{q_{ij}^{(n)}(x)\}$, respectively.

- $(B_t)_{t\geq 0}$ is a d-dim BM;
  $\kappa_1 := \max\left\{\tilde{\kappa}_0, |q_{ii}^{n,k}|; n \geq 1, 1 \leq k \leq m_n + 1\right\}$.
- $\mathcal{N}(\mathrm{d}t, \mathrm{d}z)$ is a Poisson random measure with intensity $\mathrm{d}t \times \mathrm{d}z$ supported on $[0, \infty) \times [0, \kappa_1]$;
- (B_t) and $\mathcal{N}(\mathrm{d}t, \mathrm{d}z)$ are mutually independent.

Main results–Aim-2

For any two probability measures μ and ν on $\mathbb{R}^d$, the L_1-Wasserstein distance between μ and ν is defined by

$$\mathbb{W}_1(\mu,\nu) = \inf_{\pi\in\mathscr{C}(\mu,\nu)} \Big\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|\pi(\mathrm{d}x,\mathrm{d}y) \Big\},$$

where $\mathscr{C}(\mu,\nu)$ stands for the set of all couplings of μ and ν.

(X_t, Λ_t) and $(X_t^{(n)}, \Lambda_t^{(n)})$ satisfying (9) and (10) respectively. Assume $\sigma(x,i) = \sigma \in \mathbb{R}^{d\times d}$ with determinant $\det(\sigma) > 0$,
$b(x,i) = \hat{b}(x,i) + Z(x)$,

$$|\hat{b}(x,i) - \hat{b}(y,i)| + |Z(x) - Z(y)| \le K_2|x-y|, \qquad x, y \in \mathbb{R}^d,\ i \in \mathcal{S},$$

$$\max_{i\in\mathcal{S}} \sup_{x\in\mathbb{R}^d} |\hat{b}(x,i)| \le K_2 \quad \text{for some } K_2 > 0.$$

Assume that $\exists K_3 > 0$ such that

$$|q_{ij}(x) - q_{ij}(y)| \le K_3|x-y|, x, y \in \mathbb{R}^d,\ i,j \in \mathcal{S}, \sup_{x\in\mathbb{R}^d} \max_{i\in\mathcal{S}} q_i(x) < \infty.$$

Theorem 2

Suppose $\Theta_n \to 0$, $n \to \infty$. $X_t^{(n)} \sim \mu_t^n$ and $X_t \sim \mu_t$. Then

$$\sup_{t\in[0,T]} \mathbb{W}_1(\mu_t^n, \mu_t) \le 2T^2 \mathrm{e}^{(K_2+2(N-1)K_3)T}\Theta_n, \qquad T > 0,$$

Moreover, $\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{W}_1(\mu_t^n, \mu_t) = 0$.

Lemma1

For any two Borel sets A, B in $\mathbb{R}$, denote $A\Delta B = (A\backslash B) \cup (B\backslash A)$ and $|A\Delta B|$ the Lebesgue measure of $A\Delta B$. Then, for any $i, j \in \mathcal{S}$,

$$|\Gamma_{ij}(x)\Delta\Gamma_{ij}^{(n)}(y)| \le \max_{i,j\in\mathcal{S}} |q_{ij}(x) - q_{ij}^{(n)}(y)|, \quad x, y \in \mathbb{R}^d.$$

Lemma2

It holds that

$$\frac{1}{t}\int_0^t \mathbb{P}(\Lambda_s \ne \Lambda_s^{(n)})\mathrm{d}s \le \int_0^t \mathbb{E}[\|Q(X_s) - Q^{(n)}(X_s^{(n)})\|_{\ell_1}]\mathrm{d}s, \qquad t > 0.$$

Main results–Aim-2

Proofs of Theorem2: Under the conditions,the hybrid systems (X_t, Λ_t) and $(X_t^{(n)}, \Lambda_t^{(n)})$ uniquely exist in the pathwise sense for any initial points $(x, i) \in \mathbb{R}^d \times \mathcal{S}$.

$$\mathbb{E}\Big[\sup_{0\le t\le T}|X_t^{(n)} - X_t|\Big] \le \mathbb{E}\Big[\int_0^T |b(X_s^{(n)}, \Lambda_s^{(n)}) - b(X_s, \Lambda_s)|\mathrm{d}s\Big]$$

$$\le \mathbb{E}\Big[\int_0^T |\hat{b}(X_s^{(n)}, \Lambda_s^{(n)}) - \hat{b}(X_s, \Lambda_s^{(n)})| + |Z(X_s^{(n)}) - Z(X_s)|$$

$$+ |\hat{b}(X_s, \Lambda_s^{(n)}) - \hat{b}(X_s, \Lambda_s)|\mathrm{d}s\Big]$$

$$\le K_2 \int_0^T \mathbb{E}\Big[|X_s^{(n)} - X_s| + 2\mathbf{1}_{\{\Lambda_s^{(n)} \neq \Lambda_s\}}\Big]\mathrm{d}s.$$

According to Lemma 2,

$$\int_0^T \mathbb{P}(\Lambda_s^{(n)} \neq \Lambda_s)\mathrm{d}s \le T\int_0^T \mathbb{E}\big[\|Q(X_s) - Q^{(n)}(X_s^{(n)})\|_{\ell_1}\big]\mathrm{d}s$$

$$\le T\int_0^T \Big((N-1)K_3\mathbb{E}\big[|X_s^{(n)} - X_s|\big] + \Theta_n\Big)\mathrm{d}s.$$

Hence,

$$\mathbb{E}\Big[\sup_{0\le t\le T}|X_t^{(n)}-X_t|\Big] \le 2T^2\Theta_n+\big(K_2+2(N-1)TK_3\big)\int_0^T\mathbb{E}\Big[\sup_{0\le s\le t}|X_s^{(n)}-X_s|\Big]\mathrm{d}t,$$

which implies that

$$\mathbb{E}\Big[\sup_{0\le t\le T}|X_t^{(n)}-X_t|\Big]\le 2T^2\mathrm{e}^{(K_2+2(N-1)TK_3)T}\Theta_n,$$

by Gronwall's inequality. The desired conclusion (19) follows immediately from the fact $\mathbb{W}_1(\mu_t^n,\mu_t)\le\mathbb{E}|X_t^{(n)}-X_t|$, and the proof is complete.

Note The L_1-Wasserstein distance cannot be replaced by general L_p-Wasserstein distance with $p>1$.

Mian results–Aim-3

Aim 3

Ergodicity and stability of hybrid systems with threshold type switching

Provide explicit conditions to justify the stability in probability and ergodicity of hybrid systems with threshold type switching. These conditions generalize the corresponding results for the hybrid systems with Markovian switching, and are quite sharp as being illustrated via concrete examples.

Definitions

1) The equilibrium point $x = 0$ is said to be *stable in probability* if for any $\varepsilon > 0$, $\lim_{x\to 0} \mathbb{P}\big(\sup_{t\geq 0} |X_t^{x,i}| > \varepsilon\big) = 0$ for every $i \in \mathcal{S}$;
2) The equilibrium point $x = 0$ is said to be *unstable in probability* if it is not stable in probability.
3) The equilibrium point $x = 0$ is said to be *asymptotically stable in probability* if it is stable in probability and further $\lim_{x\to 0} \mathbb{P}\big(\lim_{t\to\infty} X_t^{x,i} = 0\big) = 1$ for every $i \in \mathcal{S}$.

(X_t, Λ_t) satisfying (1), (2) and (3), whose infinitesimal generator $\mathscr{A}$ is given by

$$\mathscr{A} f(x,i) = \mathcal{L}^{(i)} f(\cdot,i)(x) + Q(x) f(x,\cdot)(i)$$

$$\mathcal{L}^{(i)} f(\cdot,i)(x) := \sum_{k=1}^{d} b_k(x,i) \frac{\partial f}{\partial x_k}(x,i) + \frac{1}{2} \sum_{k,l=1}^{d} a_{kl}(x,i) \frac{\partial^2 f}{\partial x_k \partial x_l}(x,i)$$

$$Q(x) f(x,\cdot)(i) := \sum_{j \neq i} q_{ij}(x) \big(f(x,j) - f(x,i) \big)$$

for any $f \in C_b^2(\mathbb{R}^d \times \mathcal{S})$. Here $a(x,i) = \sigma(x,i)\sigma^*(x,i)$.

(H1) $b(0,i)=0$, $\sigma(0,i)=0$ for every $i\in\mathcal{S}$. Moreover, for any sufficiently small $r_0>\varepsilon>0$, there exist $l\in\{1,2,\dots,d\}$ and $c(\varepsilon)>0$ such that $a_{ll}(x,i)>c(\varepsilon)$ for all $(x,i)\in\{x;\,\varepsilon<|x|<r_0\}\times\mathcal{S}$.

(L1) There exist constants $\beta_i\in\mathbb{R}$ for every $i\in\mathcal{S}$, a neighborhood D of 0 in $\mathbb{R}^d$, a function $\rho: D\backslash\{0\}\to(0,\infty)$ satisfying $\rho\in C^2(D\backslash\{0\})$ such that

$$\mathcal{L}^{(i)}\rho(x)\le\beta_i\rho(x),\qquad\forall\, x\in D\backslash\{0\},\ i\in\mathcal{S}.$$

(L2) There exist constants $\beta_i\in\mathbb{R}$ for every $i\in\mathcal{S}$, a neighborhood D of 0 in $\mathbb{R}^d$, functions $\rho,h: D\backslash\{0\}\to(0,\infty)$ satisfying $\rho,\,h\in C^2(D\backslash\{0\})$ such that

$$\mathcal{L}^{(i)}\rho(x)\le\beta_i h(x),\quad\forall\, x\in D\backslash\{0\},\ i\in\mathcal{S},$$

$$\lim_{x\to0}\frac{h(x)}{\rho(x)}=0,\ \lim_{x\to0}\frac{\mathcal{L}^{(i)}h(x)}{h(x)}=0.$$

Recall (3): $\Delta_m := \{0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m < \alpha_{m+1} = \infty\}$, a finite division of $[0, \infty)$,

$$q_{ij}(x) = \sum_{k=1}^{m+1} q_{ij}^{(k)} \mathbf{1}_{[\alpha_{k-1}, \alpha_k)}(|x|), \qquad i, j \in \mathcal{S},\ x \in \mathbb{R}^d$$

Theorem3-1

Let (X_t, Λ_t) be a hybrid system satisfying (1), (2), (3). Assume (H1), (A1), (A2) hold. Let $\pi^{(1)} = (\pi_i^{(1)})_{i\in\mathcal{S}}$ be the invariant probability measure of $(q_{ij}^{(1)})$. Suppose one of (L1) and (L2) holds with

$$\sum_{i\in\mathcal{S}} \pi_i^{(1)} \beta_i < 0.$$

Then the equilibrium point $x = 0$ is asymptotically stable in probability if $\rho(x)$ vanishes only at 0, and is unstable in probability if $\lim_{x\to 0} \rho(x) = \infty$.

Another complicated case in $\mathbb{R}$. $(q_{ij}^{(k)})$, $(\tilde{q}_{ij}^{(l)})$ are conservative, irreducible Q-matrices for $k \in \{1, \ldots, m_1\}$ and $l \in \{1, \ldots, m_2\}$.

$$\{-\infty = \alpha_{-m_2-1} < \ldots < \alpha_{-1} < 0 < \alpha_1 < \ldots < \alpha_{m_1} = \infty\}.$$

$$q_{ij}(x) = q_{ij}^{(1)} \mathbf{1}_{[0,\alpha_1)}(x) + \tilde{q}_{ij}^{(1)} \mathbf{1}_{(\alpha_{-1},0)}(x) + \sum_{k=2}^{m_1} q_{ij}^{(k)} \mathbf{1}_{[\alpha_{k-1},\alpha_k)}(x) + \sum_{l=1}^{m_2} \tilde{q}_{ij}^{(l)} \mathbf{1}_{(\alpha_{-l-1},\alpha_{-l}]}(x),$$

Notice that in this situation, the corresponding Markovian regime-switching processes associated respectively with the transition rate matrix $(q_{ij}^{(1)})$ and $(\tilde{q}_{ij}^{(1)})$ may own quite different stability at the equilibrium point $x = 0$.

Main results–Aim-3(1): Stability

Recall: $\{-\infty = \alpha_{-m_2-1} < \ldots < \alpha_{-1} < 0 < \alpha_1 < \ldots < \alpha_{m_1} = \infty\}$.

$$q_{ij}(x) = q_{ij}^{(1)} \mathbf{1}_{[0,\alpha_1)}(x) + \tilde{q}_{ij}^{(1)} \mathbf{1}_{(\alpha_{-1},0)}(x) + \sum_{k=2}^{m_1} q_{ij}^{(k)} \mathbf{1}_{[\alpha_{k-1},\alpha_k)} + \sum_{l=1}^{m_2} \tilde{q}_{ij}^{(l)} \mathbf{1}_{(\alpha_{-l-1},\alpha_{-l}]},$$

Theorem 3-2

Let (X_t, Λ_t) satisfy (1), (2) with $d = 1$ and $(q_{ij}(x))$ being given above. Assume (H1), (A1), (A2) hold. Denote by $(q_{ij}^{(1)}) \sim \pi^{(1)} = (\pi_i^{(1)})$ and $(\tilde{q}_{ij}^{(1)}) \sim \tilde{\pi}^{(1)} = (\tilde{\pi}_i^{(1)})$. Suppose that one of (L1) and (L2) holds.

1° If one of $\sum_{i\in\mathcal{S}} \pi_i^{(1)} \beta_i < 0$ and $\sum_{i\in\mathcal{S}} \tilde{\pi}_i^{(1)} \beta_i < 0$ holds, and $\lim_{x\to 0} \rho(x) = \infty$, then the equilibrium point $x = 0$ is unstable in probability.

2° If $\sum_{i\in\mathcal{S}} \pi_i^{(1)} \beta_i < 0$ and $\sum_{i\in\mathcal{S}} \tilde{\pi}_i^{(1)} \beta_i < 0$, and $\rho(x)$ vanishes only at 0, then the equilibrium point $x = 0$ is asymptotically stable in probability.

Main results–Aim-3(1): Stability

Consider the non-linear hybrid system $(X_t, \Lambda_t) \in \mathbb{R} \times \mathcal{S}$ satisfying

$$\mathrm{d}X_t = b_{\Lambda_t}\mathrm{sgn}(X_t)\big(|X_t|^p \wedge |X_t|\big)\mathrm{d}t + \sigma_{\Lambda_t}\big(|X_t|^q \wedge |X_t|\big)\mathrm{d}B_t,$$

where $1 < p \leq 2q - 1$, $\mathrm{sgn}(x) = 1$ if $x \geq 0$; $= -1$ if $x < 0$.

- $(\Lambda_t) \in \mathcal{S} = \{1, 2, \ldots, N\}$ satisfying (2) with $(q_{ij}(x))$ given by

  $q_{ij}(x) = q_{ij}^{(1)}\mathbf{1}_{[0,\alpha_1)}(x) + q_{ij}^{(2)}\mathbf{1}_{[\alpha_1,\infty)}(x) + \tilde{q}_{ij}^{(1)}\mathbf{1}_{(\alpha_{-1},0)}(x) + \tilde{q}_{ij}^{(2)}\mathbf{1}_{(-\infty,\alpha_{-1}]}(x),$

  where $(q_{ij}^{(k)})$, $(\tilde{q}_{ij}^{(k)})$, $k = 1, 2$ are all conservative, irreducible Q-matrices on $\mathcal{S}$; $-\infty < \alpha_{-1} < 0 < \alpha_1 < \infty$.
- Denote by $(q_{ij}^{(1)}) \sim (\pi_i^{(1)})$ and $(\tilde{q}_{ij}^{(1)}) \sim (\tilde{\pi}_i^{(1)})$.

Example1

For (X_t, Λ_t) defined above, let

$$\beta_i = \begin{cases} b_i, & p < 2q-1, \\ b_i - \frac{1}{2}\sigma_i^2, & p = 2q-1, \end{cases} \qquad i \in \mathcal{S}.$$

Then,

(1) If $\max\left\{ \sum_{i\in\mathcal{S}} \pi_i^{(1)}\beta_i, \sum_{i\in\mathcal{S}} \tilde{\pi}_i^{(1)}\beta_i \right\} < 0$, then the equilibrium point $x = 0$ is asymptotically stable in probability.

(2) If $\max\left\{ \sum_{i\in\mathcal{S}} \pi_i^{(1)}\beta_i, \sum_{i\in\mathcal{S}} \tilde{\pi}_i^{(1)}\beta_i \right\} > 0$, then $x = 0$ is unstable in probability.

- For (1), we take $\rho(x) = |x|^{\gamma}$ with $\gamma > 0$, and $h(x) = \gamma|x|^{\gamma+p-1}$.
- For (2), we take $\rho(x) = |x|^{-\gamma}$ with $\gamma > 0$, $h(x) = |x|^{p-\gamma-1}$.

Aim-3(2)

Criteria on ergodicity and transience

(L3) There exist a positive function $\rho \in C^2(\mathbb{R}^d)$, a constant $r_0 > 0$, $\beta_i \in \mathbb{R}$ for $i \in \mathcal{S}$, such that

$$\mathcal{L}^{(i)}\rho(x) \leq \beta_i \rho(x), \qquad |x| > r_0, \ i \in \mathcal{S}.$$

(L4) There are two positive functions ρ, $h \in C^2(\mathbb{R}^d)$, a constant $r_0 > 0$, constants $\beta_i \in \mathbb{R}$ for $i \in \mathcal{S}$, such that

$$\mathcal{L}^{(i)}\rho(x) \leq \beta_i h(x), \qquad |x| > r_0, \ i \in \mathcal{S},$$

$$\lim_{|x|\to\infty} \frac{h(x)}{\rho(x)} = 0, \ \lim_{|x|\to\infty} \frac{\mathcal{L}^{(i)}h(x)}{h(x)} = 0.$$

Main results–Aim-3(2): Ergodicity and transience

Recall(3): $\Delta_m := \{0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m < \alpha_{m+1} = \infty\}$, a finite division of $[0, \infty)$,

$$q_{ij}(x) = \sum_{k=1}^{m+1} q_{ij}^{(k)} \mathbf{1}_{[\alpha_{k-1}, \alpha_k)}(|x|), \qquad i, j \in \mathcal{S},\ x \in \mathbb{R}^d.$$

Theorem4-1

Let (X_t, Λ_t) be a hybrid system satisfying (1), (2), (3). Suppose (A1), (A2) hold. Let $(\pi_i^{(m+1)})$ be the invariant probability measure of $Q^{(m+1)} = (q_{ij}^{(m+1)})$. Assume that one of $(L3)$ and $(L4)$ holds, and

$$\sum_{i \in \mathcal{S}} \pi_i^{(m+1)} \beta_i < 0.$$

Then (X_t, Λ_t) is ergodic if $\lim_{|x| \to \infty} \rho(x) = \infty$; is transient if $\lim_{|x| \to \infty} \rho(x) = 0$.

Main results–Aim-3(2): Ergodicity and transience

Recall(4): $\{-\infty < \alpha_0 < \alpha_1 < \ldots < \alpha_m < \alpha_{m+1} = \infty\}$.

$$q_{ij}(x) = q_{ij}^{(0)} \mathbf{1}_{(-\infty,\alpha_0)}(x) + \sum_{k=1}^{m+1} q_{ij}^{(k)} \mathbf{1}_{[\alpha_{k-1},\alpha_k)}(x),\ i,j \in \mathcal{S},\ x \in \mathbb{R},$$

Theorem4-2

Let (X_t, Λ_t) be a hybrid system in $\mathbb{R}\times\mathcal{S}$ satisfying (1), (2) and (34). Suppose that (A1), (A2) hold and one of (L3) and (L4) holds. Let $(\pi_i^{(0)})$ and $(\pi_i^{(m+1)})$ be the invariant probability measure of $(q_{ij}^{(0)})$ and $(q_{ij}^{(m+1)})$ respectively.

(1) If $\max\Big\{\sum_{i\in\mathcal{S}} \pi_i^{(0)}\beta_i, \sum_{i\in\mathcal{S}} \pi_i^{(m+1)}\beta_i\Big\} < 0$, and $\lim_{|x|\to\infty} \rho(x) = \infty$, then (X_t, Λ_t) is ergodic.

(2) If (i) $\sum_{i\in\mathcal{S}} \pi_i^{(0)}\beta_i < 0$ and $\lim_{x\to-\infty} \rho(x) = 0$, or (ii) $\sum_{i\in\mathcal{S}} \pi_i^{(m+1)}\beta_i < 0$ and $\lim_{x\to\infty} \rho(x) = 0$, then (X_t, Λ_t) is transient.

Consider the O-U type process (X_t, Λ_t):

$$\mathrm{d}X_t = b_{\Lambda_t} X_t \mathrm{d}t + \sigma_{\Lambda_t}(X_t^2 \wedge |X_t|)\mathrm{d}B_t,$$

and (2) with $(q_{ij}(x))_{i,j\in\mathcal{S}}$ given by

$$q_{ij}(x) = q_{ij}^{(1)}\mathbf{1}_{[0,\alpha_1)}(x) + \tilde{q}_{ij}^{(1)}\mathbf{1}_{(\alpha_{-1},0)}(x) + q_{ij}^{(2)}\mathbf{1}_{[\alpha_1,\infty)}(x) + \tilde{q}_{ij}^{(2)}\mathbf{1}_{(-\infty,\alpha_{-1}]}(x),$$

where $(q_{ij}^{(1)})$, $(q_{ij}^{(2)})$, $(\tilde{q}_{ij}^{(1)})$, and $(\tilde{q}_{ij}^{(2)})$ are all conservative, irreducible Q-matrices on $\mathcal{S}$; $\alpha_{-1} < 0 < \alpha_1$.

Example2

(X_t, Λ_t) is exponentially ergodic if $\max\left\{\sum_{i\in\mathcal{S}} \pi_i^{(2)}\mu_i, \sum_{i\in\mathcal{S}} \tilde{\pi}_i^{(2)}\mu_i\right\} < 0$; (X_t, Λ_t) is transient if $\max\left\{\sum_{i\in\mathcal{S}} \pi_i^{(2)}\mu_i, \sum_{i\in\mathcal{S}} \tilde{\pi}_i^{(2)}\mu_i\right\} > 0$.

Main results–Aim-3(2): Ergodicity and transience

Let $Q(x) = (q_{ij}(x))$ be a conservative, irreducible Q-matrix on $\mathcal{S}$ for every $x \in \mathbb{R}^d$, and

$$|q_{ij}(x) - q_{ij}(y)| \le K_4|x-y|, \quad x, y \in \mathbb{R}^d,\ i, j \in \mathcal{S},$$

for some $K_4 > 0$. Assume (A1) holds. Let (X_t, Λ_t) be a hybrid system satisfying (1), (2).

Theorem4-3

Assume that the limit

$$\lim_{|x|\to\infty} Q(x) = Q \tag{11}$$

exists and Q is still irreducible. Denote by (π_i) the invariant probability measure of Q. Suppose (L3) or (L4) holds, and

$$\sum_{i\in\mathcal{S}} \pi_i\beta_i < 0. \tag{12}$$

Then (X_t, Λ_t) be expenentially ergodic if $\lim_{|x|\to\infty} \rho(x) = \infty$; is transient if $\lim_{|x|\to\infty} \rho(x) = 0$.

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Thank you!